On the Ghosts in Rutherfurd's Diffraction-Spectra.

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Let there be a periodical irregularity in the ruling of a diffraction plate, so that the side of the r^{th} slit nearest a fixed line of reference parallel to the ruling shall be distant from that line by

$$\left(r-\frac{1}{2}\alpha\right)\mathsf{w}+\mathsf{e}\sin\left(r\,\theta-\frac{1}{2}\theta\right)$$

while the side of the same opening furthest from the line of reference is distant from it by

$$\left(r+\frac{1}{2}\alpha\right)\mathsf{w}+\mathsf{e}\sin\left(r\,\theta+\frac{1}{2}\theta\right).$$

This is supposing the opaque lines to have a constant breadth, $(1-\alpha)\,\mathsf{w}$.

Suppose the collimator and telescope of the spectrometer to be focused for parallel rays, and neglect the angular aperture of the slit. Let the angle of incidence be i, and the angle of emergence j. Write

$$\nu=\sin i-\sin j.$$

Then the ray which strikes the gitter at a distance x from the line of reference is longer than that which passes through the line of reference by νx. Consequently, the resultant oscillation from the r^{th} slit will be

$$\int_{(r-\frac{1}{2}\alpha)\mathsf{w}+\mathsf{e}\sin(r\theta-\frac{1}{2}\theta)}^{(r+\frac{1}{2}\alpha)\mathsf{w}+\mathsf{e}\sin(r\theta+\frac{1}{2}\theta)}\mathrm{d}x\,.\sin 2\,\frac{Vt-\nu x}{\lambda}\,\pi$$

where t is the time, V the velocity of light, and λ the wave-length. (In this paper π will be written for the ratio of the circumference to the diameter, e for the natural base, and ι for the imaginary unit.) If then we sum this for all integral values of r, we obtain an expression for the resultant oscillation from the whole gitter.

Performing the integration relatively to x, indicating the summation relative to r, and using the abbreviations

$$\omega=2\,\frac{\mathsf{w}\nu}{\lambda}\,\pi\,,\qquad \varepsilon\omega=2\,\frac{\mathsf{e}\nu}{\lambda}\,\pi\,,\qquad \tau=2\,\frac{Vt}{\lambda}\,\pi\,,$$

we obtain the following expression for the resultant oscillation from the whole grating:

$$\begin{aligned}\frac{\mathrm{w}}{\omega}\Sigma_r\Big\{&\cos\Big[\varepsilon\omega\sin\Big(r\,\theta+\frac{1}{2}\theta\Big)\Big].\cos\Big(\tau-\frac{1}{2}\alpha\,\omega-r\,\omega\Big)\\&+\sin\Big[\varepsilon\omega\sin\Big(r\,\theta+\frac{1}{2}\theta\Big)\Big].\sin\Big(\tau-\frac{1}{2}\alpha\,\omega-r\,\omega\Big)\\&-\cos\Big[\varepsilon\omega\sin\Big(r\,\theta-\frac{1}{2}\theta\Big)\Big].\cos\Big(\tau+\frac{1}{2}\alpha\,\omega-r\,\omega\Big)\\&-\sin\Big[\varepsilon\omega\sin\Big(r\,\theta-\frac{1}{2}\theta\Big)\Big].\sin\Big(\tau+\frac{1}{2}\alpha\,\omega-r\,\omega\Big)\Big\}.\end{aligned}$$

We now need a formula for developing sines and cosines of sines. For this purpose take $y=e^{\iota x}$. Then we have

$$\cos\,(\alpha\sin x)+\sin\,(\alpha\sin x)\,.\,\iota=e^{\iota\,.\,\alpha\sin x}=e^{\frac{1}{2}\alpha\left(y-\frac{1}{y}\right)}.$$

By the usual development of an exponential function, this is

$$e^{\frac{1}{2}\alpha\left(y-\frac{1}{y}\right)}=\Sigma_p{}_0^\infty\frac{\alpha^p}{p!\,2^p}\Big(y-\frac{1}{y}\Big)^p,$$

and by the binomial theorem, this is,

$$e^{\frac{1}{2}\alpha\left(y-\frac{1}{y}\right)}=\Sigma_p{}_0^\infty\frac{\alpha^p}{p!\,2^p}\Sigma_q{}_0^p(-1)^q\frac{p!}{q!\,(p-q)!}y^{p-2q}.$$

The pq^{th} term is

$$(-1)^q\frac{\alpha^p y^{p-2q}}{2^p q!\,(p-q)!}.$$

Put $m=p-2q$ and this becomes

$$(-1)^q\frac{\alpha^m y^m}{2^m}\cdot\frac{\alpha^{2q}}{4^q q!\,(m+q)!}.$$

In regard to the limits of the summation, q may have any value from zero to positive infinity, and, for every value of q, p may have any value from q to positive infinity; hence, m may have any value from $-q$ to positive infinity, and we have

$$\cos(\alpha\sin x)+\sin\,(\alpha\sin x)\,.\,\iota=\Sigma_q{}_0^\infty(-1)^q\frac{\alpha^{2q}}{4^q.\,q!}\Sigma_m{}_{-q}^\infty\frac{\alpha^m}{2^m.\,(m+q)!}(\cos mx+\sin mx\,.\,\iota).$$

If m has a positive value, q may have any positive value; but if m has a negative value, q can only have any positive value greater than $-m$. Hence, we may take the terms for which m is not zero in pairs, embracing in each pair a term for which m has a positive value, M, and q has a value, Q, and also a term for which $m=-M$ and $q=M+Q$. The sum of two terms composing the pair is, then,

$$\begin{aligned}&(-1)^Q\frac{\alpha^M(\cos Mx+\sin Mx\,.\,\iota)}{2^M}\cdot\frac{\alpha^{2Q}}{4^Q Q!\,(M+Q)!}\\&+(-1)^{M+Q}\frac{\alpha^{-M}(\cos Mx-\sin Mx\,.\,\iota)}{2^{-M}}\cdot\frac{\alpha^{2M+2Q}}{4^{M+Q}(M+Q)!\;Q!}.\end{aligned}$$

If M is even, the value of this is

$$(-1)^Q \frac{a^M}{2^{M-1}} \frac{a^{2Q}}{4^Q Q!\,(M+Q)!} \cos Mx\,;$$

and if M is odd, its value is

$$(-1)^Q \frac{a^M}{2^{M-1}} \frac{a^{2Q}}{4^Q Q!\,(M+Q)!} \sin Mx\,.\,\iota.$$

We have then

$$\cos(\alpha \sin x) + \sin(\alpha \sin x)\,.\,\iota = \Sigma_{q\,0}^{\,\infty} (-1)^q \frac{a^{2q}}{4^q (q\,!)^2} + \Sigma_{m\,1}^{\,\infty} \frac{A_m a^m}{m!\,2^{m-1}} (\cos x + \sin x\,.\,\iota)^m\,;$$

where

$$A_m = \Sigma_{q\,0}^{\,\infty} (-1)^q \frac{m\,!}{4^q q\,!\,(m+q)!}\, a^{2q}.$$

Performing the numerical calculations, we have

$$\begin{aligned}
\cos(\alpha \sin x) = & \left(1 - \frac{1}{4}\alpha^2 + \frac{1}{64}\alpha^4 - \frac{1}{2304}\alpha^6 + \frac{1}{147456}\alpha^8 - \frac{1}{14745600}\alpha^{10} + \text{etc.}\right) \\
& + \frac{1}{4}\alpha^2 \left(1 - \frac{1}{12}\alpha^2 + \frac{1}{384}\alpha^4 - \frac{1}{23040}\alpha^6 + \frac{1}{2211840}\alpha^8 - \text{etc.}\right) \cos 2x \\
& + \frac{1}{192}\alpha^4 \left(1 - \frac{1}{20}\alpha^2 + \frac{1}{960}\alpha^4 - \frac{1}{80640}\alpha^6 + \text{etc}\right) \cos 4x \\
& + \frac{1}{23040}\alpha^6 \left(1 - \frac{1}{28}\alpha^2 + \frac{1}{1792}\alpha^4 - \text{etc.}\right) \cos 6x \\
& + \frac{1}{5160960}\alpha^8 \left(1 - \frac{1}{36}\alpha^2 + \text{etc.}\right) \cos 8x \\
& + \frac{1}{1857945600}\alpha^{10} (1 - \text{etc.}) \cos 10x \\
& + \text{etc.}
\end{aligned}$$

$$\begin{aligned}
\sin(\alpha \sin x) = \alpha & \left(1 - \frac{1}{8}\alpha^2 + \frac{1}{192}\alpha^4 \frac{1}{9216}\alpha^6 + \frac{1}{737280}\alpha^8 - \frac{1}{88473600}\alpha^{10} + \text{etc.}\right) \sin x \\
& + \frac{1}{24}\alpha^3 \left(1 - \frac{1}{16}\alpha^2 + \frac{1}{640}\alpha^4 - \frac{1}{46080}\alpha^6 + \frac{1}{5160960}\alpha^8 - \text{etc.}\right) \sin 3x \\
& + \frac{1}{1920}\alpha^5 \left(1 - \frac{1}{24}\alpha^2 + \frac{1}{1344}\alpha^4 - \frac{1}{129024}\alpha^6 + \text{etc.}\right) \sin 5x \\
& + \frac{1}{322560}\alpha^7 \left(1 - \frac{1}{32}\alpha^2 + \frac{1}{2304}\alpha^4 - \text{etc.}\right) \sin 7x \\
& + \frac{1}{92897280}\alpha^9 \left(1 - \frac{1}{40}\alpha^2 + \text{etc.}\right) \sin 9x \\
& + \frac{1}{40874803200}\alpha^{11} (1 - \text{etc.}) \sin 11x \\
& + \text{etc.}
\end{aligned}$$

Making use of these series, the expression for the resultant oscillation from the gitter becomes

$$-\mathrm{w}\sum_0^\infty{}_{(\text{even } m)} A_m \frac{\epsilon^m \omega^{m-1}}{m!\,2^{m-2}}\,\Sigma_r\Big(\cos mr\theta\,.\,\sin(r\omega-\tau)\,.\,\cos\frac{1}{2}m\theta\,.\,\sin\frac{1}{2}\alpha\omega$$
$$+\sin mr\theta\,.\,\cos(r\omega-\tau)\,.\,\sin\frac{1}{2}m\theta\,.\,\cos\frac{1}{2}\alpha\omega\Big)$$
$$-\mathrm{w}\sum_1^\infty{}_{(\text{odd } m)} A_m \frac{\epsilon^m \omega^{m-1}}{m!\,2^{m-2}}\,\Sigma_r\Big(\cos mr\theta\,.\,\sin(r\omega-\tau)\,.\,\sin\frac{1}{2}m\theta\,.\,\cos\frac{1}{2}\alpha\omega$$
$$+\sin mr\theta\,.\,\cos(r\omega-\tau)\,.\,\cos\frac{1}{2}m\theta\,.\,\sin\frac{1}{2}\alpha\omega\Big).$$

The summation relatively to r may be effected by means of the formula,

$$\Sigma_x \sin(hx+a)\,.\,\sin(kx+b)=$$
$$\frac{-\sin\left(hx+a-\frac{1}{2}h\right).\cos\left(kx+b-\frac{1}{2}k\right).\cos\frac{1}{2}h.\sin\frac{1}{2}k+\cos\left(hx+a-\frac{1}{2}h\right).\sin\left(kx+b-\frac{1}{2}k\right).\sin\frac{1}{2}h.\cos\frac{1}{2}k.}{\cos h-\cos k}$$

For a modern gitter, it would be quite as satisfactory to consider r as infinite, and to use, in place of the above, an infinitesimal formula, which will be found in Hirsch's Integral Tables. Applying, however, the formula of finite integration, we have, as an integrated expression for the resultant oscillation from the whole gitter,

$$\frac{\mathrm{w}}{\omega}\,\frac{A_0}{1-\cos\omega}\cos\left(r\omega-\tau-\frac{1}{2}\omega\right)\left[\cos\frac{1}{2}(\omega-\alpha\omega)-\cos\frac{1}{2}(\omega+\alpha\omega)\right]$$
$$+\mathrm{w}\sum_2^\infty{}_{\text{even } m)} A_m \frac{\dfrac{\epsilon^m\omega^{m-1}}{m!\,2^{m-1}}}{\cos m\theta-\cos\omega}\Big\{-\sin m\left(r\theta-\frac{1}{2}\theta\right).\sin\left(r\omega-\tau-\frac{1}{2}\omega\right).\sin m\theta.\sin\frac{1}{2}(\omega-\alpha\omega)$$
$$+\cos m\left(r\theta-\frac{1}{2}\theta\right).\cos\left(r\omega-\tau-\frac{1}{2}\omega\right)\left[\cos m\theta\,.\,\cos\frac{1}{2}(\omega-\alpha\omega)-\cos\frac{1}{2}(\omega+\alpha\omega)\right]\Big\}$$
$$+\mathrm{w}\sum_1^\infty{}_{(\text{odd } m)} A_m \frac{\dfrac{\epsilon^m\omega^{m-1}}{m!\,2^{m-1}}}{\cos m\theta-\cos\omega}\Big\{\cos m\left(r\theta-\frac{1}{2}\theta\right).\cos\left(r\omega-\tau-\frac{1}{2}\omega\right).\sin m\theta.\sin\frac{1}{2}(\omega-\alpha\omega)$$
$$-\sin m\left(r\theta-\frac{1}{2}\theta\right).\sin\left(r\omega-\tau-\frac{1}{2}\omega\right)\left[\cos m\theta\,.\,\cos\frac{1}{2}(\omega-\alpha\omega)-\cos\frac{1}{2}(\omega+\alpha\omega)\right]\Big\}.$$

This expression may be simplified by writing

$$x=\frac{1}{2}(\omega+m\theta)\,,$$
$$y=\frac{1}{2}(\omega-m\theta)\,;$$

so that

$$\sin\left[\left(r-\frac{1}{2}\right)m\theta\right].\sin\left[\left(r-\frac{1}{2}\right)\omega-\tau\right]=\frac{1}{2}\cos\left[(2r-1)y-\tau\right]-\frac{1}{2}\cos\left[(2r-1)x-\tau\right]$$

$$\cos\left[\left(r-\frac{1}{2}\right)m\theta\right].\cos\left[\left(r-\frac{1}{2}\right)\omega-\tau\right]=\frac{1}{2}\cos\left[(2r-1)y-\tau\right]+\frac{1}{2}\cos\left[(2r-1)x-\tau\right].$$

We have also to observe that

$$\mp\sin m\theta\,.\sin\frac{1}{2}(\omega-\alpha\omega)+\cos m\theta\,.\cos\frac{1}{2}(\omega-\alpha\omega)-\cos\frac{1}{2}(\omega+\alpha\omega)$$

$$=\cos\left[\frac{1}{2}(\omega-\alpha\omega)\pm m\theta\right]-\cos\frac{1}{2}(\omega+\alpha\omega)=+2\sin\frac{1}{2}(\omega\pm m\theta)\sin\frac{1}{2}(\alpha\omega\mp m\theta).$$

Thus, the quantity in parenthesis, under the sum for even values of m, reduces to

$$\cos\left[(2r-1)\,y-\tau\right].\sin\frac{1}{2}(\omega+m\theta)\,.\sin\frac{1}{2}(\alpha\omega-m\theta)$$

$$+\cos\left[(2r-1)\,x-\tau\right].\sin\frac{1}{2}(\omega-m\theta)\,.\sin\frac{1}{2}(\alpha\omega+m\theta),$$

and the corresponding quantity for odd values of m, to

$$-\cos\left[(2r-1)\,y-\tau\right].\sin\frac{1}{2}(\omega+m\theta).\sin\frac{1}{2}(\alpha\omega-m\theta)$$

$$+\cos\left[(2r-1)\,x-\tau\right].\sin\frac{1}{2}(\omega-m\theta)\,.\sin\frac{1}{2}(\alpha\omega+m\theta).$$

The integral is to be taken between limiting values of r, say r_1 and r_2. Let the whole number of openings in the gitter be R, so that

$$R=r_2-r_1\,.$$

Then, a second equation to determine r_1 and r_2 may be assumed arbitrarily without affecting the result. Let this equation be

$$r_2+r_1=1\,.$$

Then

$$(2r_2-1)=-(2r_1-1)=R\,.$$

Now r occurs only in the factors

$$\cos\left[(2r-1)\;y-\tau\right]=\cos\,(2r-1)\;y\,.\cos\tau+\sin\,(2r-1)\;y\,.\sin\tau$$

and

$$\cos\left[(2r-1)\;x-\tau\right]=\cos\,(2r-1)\;x\,.\cos\tau+\sin\,(2r-1)\;x\,.\sin\tau\,.$$

Taken between these limits, these factors will be respectively,

$$2\sin Ry\,.\sin\tau\,,$$

$$2\sin Rx\,.\sin\tau\,.$$

Applying these reductions, and also remembering that

$$\cos m\theta-\cos\omega=2\sin x\sin y\,,$$

the expression for the resultant oscillation from the whole gitter reduces to

$$\sin \tau \,.\, \mathrm{w} \sum_{-\infty}^{+\infty}{}_m A_m \frac{\epsilon^m \omega^{m-1}}{m!\ 2^{m-1}} \frac{\sin \frac{1}{2} R(\omega + m\theta)}{\sin \frac{1}{2} (\omega + m\theta)} \sin \frac{1}{2} (\alpha\omega + m\theta)\,,$$

where, in summing for negative values of m, positive values are to be taken in the coefficients, and where terms arising from odd negative values of m in the parenthesis are to have the opposite sign, and where the term in $m = 0$ is to have only half the above value.

We have now to study the principal maxima of the amplitude of this oscillation, for varying ω. Taking each term of the series separately, we observe that one factor of it, namely,

$$\frac{\sin \frac{1}{2} R(\omega + m\theta)}{\sin \frac{1}{2} (\omega + m\theta)},$$

reaches a maximum when

$$\omega + m\theta = 2N\pi\,,$$

and this maximum value is R. Now R is a number amounting to several thousand, while α is less than unity. Hence, the maximum of the whole term will be very nearly at the same place, and one of the maxima of the sum of all the terms will also be nearly in that place.

To ascertain the precise position of the maximum of any one term, put

$$\omega = 2N\pi - m\theta + \delta\omega\,.$$

Then, neglecting the cube of $\delta\omega$, in comparison with unity, we have

$$\sin \frac{1}{2} R\,(\omega + m\theta) = \pm \sin \frac{1}{2} R\delta\omega = \pm \frac{1}{2} R\delta\omega \mp \frac{1}{48} R^3\,(\delta\omega)^3$$

$$\sin \frac{1}{2} (\omega + m\theta) = \pm \sin \frac{1}{2} \delta\omega = \pm \frac{1}{2} \delta\omega \mp \frac{1}{48} (\delta\omega)^3$$

$$\frac{\sin \frac{1}{2} R(\omega + m\theta)}{\sin \frac{1}{2} (\omega + m\theta)} = \pm \frac{\sin \frac{1}{2} R\delta\omega}{\sin \frac{1}{2} \delta\omega} = \pm R \mp \frac{1}{24} (R^3 - R)\,(\delta\omega)^2.$$

As for $\sin \frac{1}{2} (\alpha\omega + (-1)^m m\theta)$, it may have any value whatever from -1 to $+1$, according to the magnitude of α. But it is when it vanishes that the maximum is at the greatest value of $\delta\omega$. Let us then suppose

$$\sin \frac{1}{2} (\alpha\omega + (-1)^m m\theta) = \pm \frac{1}{2} \alpha\delta\omega \mp \frac{1}{48} \alpha^3\,(\delta\omega)^3.$$

Finally, there is the factor ω^{m-1}. Dividing this by $(2N\pi - m\theta)^{m-1}$, we have $\left(\frac{\omega}{2N\pi - m\theta}\right)^{m-1} = 1 + (m-1)(2N\pi - m\theta)^{-1}\,\delta\omega + \frac{(m-1)(m-2)}{2}(2N\pi - m\theta)^{-2}(\delta\omega)^2$; finally, multiplying together the quantities thus obtained, we find as that factor of the mth term which contains $(\delta\omega)$

$$\delta\omega + (m-1)(2N\pi - m\theta)^{-1}\,\delta\omega^2 + \left\{\frac{(m-1)(m-2)}{2}(2N\pi - m\theta)^{-2} - \frac{1}{24}\alpha^2 - \frac{1}{24}(R^2-1)\right\}(\delta\omega)^3.$$

Differentiating, we find as the equation for determining the value of $\delta\omega$ at the maximum of the mth term

$$1 + 2(m-1)(2N\pi - m\theta)^{-1}\,\delta\omega + 3\left\{\frac{(m-1)(m-2)}{2}(2N\pi - m\theta)^{-2} - \frac{1}{24}\alpha^2 - \frac{1}{24}(R^2-1)\right\}(\delta\omega)^2 = 0.$$

If we neglect $\frac{1}{R^2}$, the solution of this equation is

$$\delta\omega = \frac{8(m-1)}{R^2(2N\pi - m\theta)}.$$

It will be seen that $\delta\omega$ is zero when $m = 1$, and that for the principal spectrum, for which $m = 0$, if $R = 1000$, $\frac{\delta\omega}{\omega}$ is altogether inappreciable, but if $R = 100$, $\frac{\delta\omega}{\omega} =$ about $\frac{1}{50000}$ for the first order, which displaces the spectrum by about $\frac{1}{50}$ part of the distance between the two D lines.

We have now to consider how far the maxima of the sum of the series representing the oscillation may differ from those of the single terms. A term will have the most influence in displacing a maximum when it is itself nearly zero, or more accurately when its differential coefficient relatively to ω is at a maximum. As ω increases by 2π so as to pass from one principal maximum of oscillation to another, $R\omega$ passes R times through 2π, so that the term passes through as many maxima and minima. Then the differential coefficient relative to ω of the sum of all the terms will be the greatest for a value of ω such that

$$\omega + m_0\theta = 2N\pi,$$

(m_0 being a given value of m), when, in addition to the above equation, we have

$$R\theta = 4N\pi.$$

In this case, the differential coefficient of the mth term of the expression for the oscillation will be

$$\frac{R}{\omega} m!\left(\frac{\epsilon\omega}{2}\right)^2 \frac{1}{\sin\frac{1}{2}(\omega + m\theta)}.$$

It will be sufficiently accurate to put

$$\sin\frac{1}{2}(\omega + m\theta) = \frac{1}{2}(m - m_0)\,\theta.$$

Then it is plain that, were the term for $m = 0$ of the same value as the others, the total differential coefficient would be

$$\frac{R}{\omega} m_0\, e^{\left(\frac{\epsilon\omega}{2}\right)}$$

Owing, however, to the term for $m = 0$ having only half the value given by the formula, the value is

$$\frac{R}{\omega} m_0 \left(e^{\left(\frac{\epsilon\omega}{2}\right)} - \frac{1}{2}\right).$$

In consequence of the differential coefficient having this value, the maximum will not occur exactly at the value of α for which

$$\omega + m_0\theta = 2N\pi,$$

but will be shifted along to the point where the differential coefficient of the m_0th term is equal to the negative of the differential coefficient just found. If $\delta\omega$ is the amount of the shifting, the m_0th term of the oscillation (R being very large) is

$$\frac{\sin.\frac{R}{2}\delta\omega}{\delta\omega}.$$

The differential coefficient of this is

$$\frac{1}{4}\,\frac{\sin.R.\delta\omega - R\delta\omega}{(\delta\omega)^2},$$

and the equation to determine $\delta\omega$ is

$$\frac{1}{4}\,\frac{\sin.R\delta\omega - R\delta\omega}{(\delta\omega)^2} = \frac{R}{\omega} m_0 \left(1 - e^{\frac{\epsilon\omega}{2}}\right).$$

In the worst case, this becomes

$$\delta\omega = \frac{24}{R^2} m_0 \left(e^{\frac{\epsilon\omega}{2}} - 1\right).$$

It thus appears that the position of the principal spectrum will not be disturbed by the circumstance here considered, and that the distance between the successive ghosts will be very slightly altered.

It is to be remarked that, when two spectral lines fall very near together, they will be attracted to one another in consequence of the mixture of light

by a sensible amount. This will especially affect the position of a faint line near a very intense one.

The Phenomena.

Mr. Rutherfurd's diffraction-plates are ruled with a machine which is described by Professor A. M. Mayer in the article "Spectrum," in the second edition of *Appleton's Cyclopædia.* In consequence of the periodic error of the screw, a periodic inequality is produced in the ruling. This is shewn by putting a gitter into the spectrometer, illuminating it with homogeneous light, and observing it without the eye-piece, when it appears striped. If the eye-piece is replaced and a real solar spectrum is thrown on the slit-plate, of such purity that the light admitted into the slit varies only by a few ten-thousandths of a micron in wave-length, the maxima of light which have been investigated above appear as repetitions of the principal spectrum, in which even the fine lines due to the solar atmosphere are distinctly visible.

The positions of some of these "ghosts," or repetitions of the principal spectrum, have been carefully measured in order to test the theory.

Measures of the Positions of the Ghosts.

To determine whether the screw of the filar micrometer had the same pitch throughout its length, the distance between D_1 and D_2 was measured on different places on the screw. Gitter: speculum metal 681 lines to the millimeter. Second order, principal spectrum. Readings given are means of five pointings each. Date: 1879, July 3.

Place on the Screw	First End.		Second End.		Second End.		First End.	
Line of Spectrum	D	D_2	D_1	D_2	D_2	D_1	D_2	D_1
Micrometer reading	7^{r} 109	$7^{r}.947$	$12^{r}.108$	$12^{r}.943$	$12^{r}.937$	$12^{r}.102$	$7^{r}.925$	$7^{r}.089$
Distance of Lines	$0^{r}.838$		$0^{r}.835$		$0^{r}.835$		$0^{r}.836$	

The following were made with a speculum-metal gitter of 340½ teeth to the millimeter. Each reading given is the mean of five pointings. Date: 1879, July 3. To pass from one spectrum to another the gitter alone was turned.

Order of Spectrum	Order IV.						
Number of Ghost	Ghost, − 1.		Ghost, 0.		Ghost, + 1.		Means.
Line of Spectrum	D_2	D_1	D_2	D_1	D_2	D_1	
Micrometer reading	$8^{r}.241$	$9^{r}.330$	$9^{r}.723$	$10^{r}.800$	$11^{r}.187$	$12^{r}.272$	
Distance ($D_1 - D_2$)	$1^{r}.089$		$1^{r}.077$		$1^{r}.085$		$1^{r}.084$
Distance of successive Ghosts D_2		$1^{r}.482$		$1^{r}.464$			1 .473
Distance of successive Ghosts D_1		1 .470		1 .472			1 .471
Mean		1 .476		1 .468			1 .472

Order of Spectrum	Order V.						
Number of Ghost	Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Means.
Line of Spectrum	D_2	D_1	D_2	D_1	D_2	D_1	
Micrometer reading	$7^r.847$	$9^r.337$	$9^r.466$	$10^r.962$	$11^r.090$	$12^r.575$	
Distance ($D_1 - D_2$)	$1^r.490$		$1^r.496$		$1^r.485$		$1^r.490$
Distance of successive Ghosts D_2		$1^r.619$		$1^r.624$			1 .621
Distance of successive Ghosts D_1		1 .625		1 .613			1 .619
Mean		1 .622		1 .618			1 .620

Order of Spectrum	Order VI.						
Number of Ghost	Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Means.
Line of Spectrum	D_2	D_1	D_2	D_1	D_2	D_1	
Micrometer reading	$7^r.378$	$9^r.421$	$9^r.265$	$11^r.304$	$11^r.152$	$13^r.173$	
Distance ($D_1 - D_2$)	$2^r.043$		$2^r.039$		$2^r.021$		$2^r.034$
Distance of successive Ghosts D_2		$1^r.887$		$1^r.887$			1 .887
Distance of successive Ghosts D_1		1 .883		1 .869			1 .876
Mean		1 .885		1 .878			1 .881

Order of Spectrum	Order VII.						
Number of Ghost	Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Means.
Line of Spectrum	D_2	D_1	D_2	D_1	D_2	D_1	
Micrometer reading	$6^r.637$	$9^r.595$	$8^r.955$	$11^r.876$	$11^r.262$	$14^r.191$	
Distance ($D_1 - D_2$)	$2^r.958$		$2^r.921$		$2^r.929$		$2^r.936$
Distance of successive Ghosts D_2		$2^r.318$		$2^r.307$			2 .312
Distance of successive Ghosts D_1		2 .281		2 .315			2 .298
Mean		2 .299		2 .311			2 .305

Order of Spectrum	Order VIII.						
Number of Ghost	Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Means.
Line of Spectrum	D_2	D_1	D_2	D_1	D_2	D_1	
Micrometer reading	$4^r.737$	$9^r.467$	$8^r.002$	$12^r.680$	$11^r.256$	$15^r.885$	
Distance ($D_1 - D_2$)	$4^r.730$		$4^r.678$		$4^r.629$		$4^r.679$
Distance of successive Ghosts D_2		$3^r.265$		$3^r.254$			3 .261
Distance of successive Ghosts D_1		3 .213		3 .205			3 .209
Mean		3 .239		3 .229			3 .234

Order of Spectrum	Order IX.						
Number of Ghost	Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Means.
Line of Spectrum	D_2	D_1	D_2	D_1	D_2	D_1	
Micrometer reading	$\bar{6}^r.865$*	$9^r.403$	$4^r.281$	$16^r.977$	$12^r.075$	$24^r.435$	
Distance ($D_1 - D_2$)	$12^r.538$		$12^r.696$		$12^r.360$		$12^r.532$
Distance of successive Ghosts D_2		$7^r.416$		$7^r.794$			7 .605
Distance of successive Ghosts D_1		7 .574		7 .458			7 .516
Mean		7 .495		7 .626			7 .560

* Read $\bar{5}^r.865$. Either this is an erroneous reading, or a wrong line was measured.

The following measures were made with a metal gitter of 681 lines to the millimeter. Dates: 1879, June 20 and July 2.

Order of Spectrum	Order I.									
Number of Ghost	Ghost, — 2.		Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Ghost, + 2.	Means.
Line of Spectrum	D_2 D_1		D_2 D_1		D_2 D_1		D_2 D_1		D_2 D_1	
Micrometer reading	$7^r.286$ $7^r.799$		$8^r.632$ $9^r.112$		$9^r.925$ $10^r.383$		$11^r.196$ $11^r.664$		$13^r.496$ $12^r.928$	
$D_1 - D_2$	$0^r.513$		$0^r.480$		$0^r.458$		$0^r.468$		$0^r.432$	$0^r.470$
Distance of successive Ghosts D_2		$1^r.346$		$1^r.293$		$1^r.271$		$1^r.300$		1 .302
Distance of successive Ghosts D_1		1 .313		1 .271		1 .281		1 .264		1 .282
Mean		1. 330		1 .282		1 .276		1 .282		1 .292

Order of Spectrum	Order II.									
Number of Ghost	Ghost, — 2.		Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Ghost, + 2.	Means.
Line of Spectrum	D_2 D_1		D_2 D_1		D_2 D_1		D_2 D_1		D_2 D_1	
Micrometer reading	$5^r.312$ $2^r.482$		$6^r.907$ $8^r.059$		$8^r.477$ $9^r.627$		$10^r.067$ $11^r.191$		$11^r.632$ $12^r.752$	
$D_1 - D_2$	$1^r.170$		$1^r.152$		$1^r.150$		$1^r.124$		$1^r.120$	$1^r.143$
Distance of successive Ghosts D_2		$1^r.595$		$1^r.570$		$1^r.590$		$1^r.565$		1 .580
Distance of successive Ghosts D_1		1 .577		1 .568		1 .564		1 .561		1 .568
Mean		1 .586		1 .569		1 .577		1 .563		1 .574

Order of Spectrum	Order III.									
Number of Ghost	Ghost, — 2.		Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Ghost, + 2.	Means.
Line of Spectrum	D_2 D_1		D_2 D_1		D_2 D_1		D_2 D_1		D_2 D_1	
Micrometer reading	$4^r.593$ $6^r.896$		$6^r.713$ $9^r.053$		$8^r.876$ $11^r.205$		$10^r.989$ $13^r.280$		$13^r.057$ $15^r.308$	
$D_1 - D_2$	$2^r.303$		$2^r.340$		$2^r.329$		$2^r.291$		$2^r.251$	$2^r.303$
Distance of successive Ghosts D_2		$2^r.120$		$2^r.163$		$2^r.113$		$2^r.068$		2 .116
Distance of successive Ghosts D_1		2 .157		2 .152		2 .075		2 .028		2 .103
Mean		2 .138		2 .158		2 .094		2 .048		2 .110

The following measures were made on spectra produced by a narrow silvered-glass plate of 681 lines to the millimeter. This gitter was selected as making unusually bright ghosts. The refraction by the glass must sensibly displace the ghosts. The two sodium lines, and the nickel line between them, were observed. Date: 1879, June 19.

Order of Sp.	ORDER I.									
No. of Ghost	Ghost, — 2.		Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Ghost, + 2.	Means.
Line of Sp.	D_2	D_1	D_2	D_1	D_2	D_1	D_2	D_1		
Mic. reading	$9^r.076$	$9^r.590$	$10^r.405$	$10^r.886$	$11^r.695$	$12^r.164$	$12^r.976$	$13^r.439$		
$D_1 - D_2$	$0^r.513$		$0^r.480$		$0^r.469$		$0^r.463$			$0^r.481$
Dist. suc. Ghosts D_2		$1^r.329$		$1^r.290$		$1^r.281$				1 .300
Dist. suc. Ghosts D_1		1 .296		1 .279		1 .275				1 .283
Mean		1 .312		1 .284		1 .278				1 .291

Order of Sp.	ORDER II.															
No. of Ghost	Ghost, — 2.			Ghost, — 1.			Ghost, 0.			Ghost, + 1.			Ghost, + 2.			Means.
Line of Sp.	D_2		D_1	D_2	Ni	D_1	D_2	Ni	D_1	D_2	Ni	D_1	D_2	Ni	D_1	
Mic. reading	$5^r.451$		$6^r.531$	$6^r.916$	$7^r.465$	$7^r.981$	$8^r.370$	$8^r.926$	$9^r.439$	$9^r.838$	$10^r.379$	$10^r.897$	$11^r.289$	$11^r.837$	$12^r.345$	
Ni — D_2		$1^r.080$			$0^r.549$			$0^r.556$			$0^r.541$			$0^r.548$		$0^r.548$
D_1 — Ni						$0^r.516$			$0^r.513$			$0^r.518$			$0^r.508$	0 .514
Dist. success. Ghosts D_2			$1^r.465$			$1^r.454$			$1^r.468$			$1^r.451$				1 .460
Dist. success. Ghosts Ni						1 .461			1 .453			1 .458				1 .457
Dist. success. Ghosts D_1			1 .450			1 .458			1 .458			1 .448				1 .454
Mean			1 .458			1 .458			1 .460			1 .452				1 .457

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Order of Sp.	ORDER III.															
No. of Ghost	Ghost, — 2.			Ghost, — 1.			Ghost, 0.			Ghost, + 1.			Ghost, + 2.			Means.
Line of Sp.	D_2	Ni	D_1	D_2	Ni	D_1	D_2	Ni	D_1	D_2	Ni	D_1	D_2	Ni	D_1	
Mic. reading	$6^r.822$	$7^r.874$	$8^r.843$	$8^r.671$	$9^r.715$	$10^r.693$	$10^r.515$	$11^r.559$	$12^r.535$	$12^r.365$	$13^r.399$	$14^r.372$	$14^r.192$	$15^r.228$	$16^r.198$	
Ni — D_2		$1^r.052$			$1^r.044$			$1^r.044$			$1^r.034$			$1^r.036$		$1^r.044$
D_1 — Ni			$0^r.969$			$0^r.978$			$0^r.976$			$0^r.973$			$0^r.970$	0 .973
Dist. success. Ghosts D_2			$1^r.849$			$1^r.844$			$1^r.850$			$1^r.827$				1 .842
Dist. success. Ghosts Ni			1 .841			1 .844			1 .840			1 .829				1 .838
Dist. success. Ghosts D_1			1 .850			1 .842			1 .837			1 .826				1 .839
Mean			1 .847			1 .843			1 .842			1 .827				1 .840

Order of Sp.	ORDER IV.										
No. of Ghost	Ghost, — 2.		Ghost, — 1.		Ghost, 0.		Ghost, + 1.		Ghost, + 2.		Means.
Line of Sp.	D_2	D_1	D_2	D_1	D_2	D_1	D_2	D_1	D_2	D_1	
Mic. reading	$3^r.614$	$8^r.222$	$6^r.777$	$11^r.361$	$9^r.939$	$14^r.473$	$13^r.064$	$17^r.581$	$16^r.148$	$20^r.616$	
D_1 — D_2	$4^r.608$		$4^r.584$		$4^r.534$		$4^r.517$		$4^r.468$		$4^r.542$
Dist. success. Ghosts D_2		$3^r.163$		$3^r.162$		$3^r.125$		$3^r.084$			3 .134
Dist. success. Ghosts D_1		3 .139		3 .112		3 .108		3 .035			3 .096
Mean		3 .151		3 .137		3 .116		3 .060			3 .115

The following measures were made upon C, with the metal gitter of 681 lines per mm. The distance of the fine line $\lambda = 6567.91$ (Å.) from C was measured in the principal spectrum to determine the dispersion. Date: 1879, July 1.

Order I.

Ghost, − 1.		Ghost, 0.		Ghost, + 1.	Fine line.		C.
$8^r.241$		$9^r.792$		$11^r.289$	$9^r.255$		$9^r.801$
	$1^r.551$		$1^r.497$			$0^r.546$	

Order II.

Ghost, − 1.		Ghost, 0.		Ghost, + 1.	Fine line		C.
$8^r.054$		$9^r.941$		$11^r.774$	$8^r.629$		$9^r.960$
	$1^r.887$		$1^r.833$			$1^r.331$	

Order III.

Ghost, − 1.		Ghost, 0.		Ghost, + 1.	Fine line.	
$7^r.115$		$10^r.010$		$12^r.734$	$7^r.054$	
	$2^r.895$		$2^r.724$			$2^r.956$

The following measure was made upon F, with the same gitter. The mean of lines 4870.47 and 4871.29 was pointed on to determine the dispersion. Date: 1879, July 1,

Order II.

Double.			F.	
Ghost, 0.		Ghost, 0.		Ghost, + 1.
$8^r.617$		$10^r.484$		$11^r.683$
	$1^r.867$		$1^r.190$	

The above measures satisfy the theory moderately well. Thus, according to theory, the product of the ratio of the distance of successive ghosts to the distance between the D line by the order of the spectrum should be constant, and should be twice as great for the gitter of 340½ lines to the millimeter as for that of 681 lines to the millimeter. Now this product is as follows:

Metal Gitter of 340½ *lines to the mm.*

Order	IV.	$5.43 = 2 \times 2.72$
"	V.	$5.44 = 2 \times 2.72$
"	VI.	$5.55 = 2 \times 2.77$
"	VII.	$5.50 = 2 \times 2.75$
"	VIII.	$5.53 = 2 \times 2.76$
"	IX.	$5.46 = 2 \times 2.73$

Metal Gitter of 681 *lines to the mm.*

Order	I.	2.75
"	II.	2.75
"	III.	2.75

Silvered-glass Gitter of 681 *lines to the mm.*

Order	I.	2.68
"	II.	2.74
"	III.	2.74
"	IV.	2.74

It is evident that the value which best satisfies the observations lies between 2.74 and 2.75. This ratio multiplied by the ratio of the difference of wave-length of the D lines to their mean wave-length, should give the number of lines of the finer gitters to a period of the inequality. This, from the construction of the ruling-machine, is known to be nearly, but not exactly, 360. Mr. Chapman, who works with the machine, has made certain observations, from which it would appear that the period differs about 1 per cent. from 360. The product of the ratios just mentioned (taking 2.746 for the first) is 357. This is therefore a happy confirmation of the theory.

Next, using the value 2.746, I calculate by least squares the best values of the distance of the D lines and the distance of consecutive ghosts in each order. In this way, we shall be able to judge whether the discrepancies of the observations from theory are, or are not, greater than their probable errors. The results are as follows:

Metal Gitter of $340\frac{1}{2}$ *lines to the mm.*

	Distance $D_1 - D_2$.			Distance of successive Ghosts.		
Order.	Obs.	Calc.	O. — C.	Obs.	Calc.	O. — C.
IV.	1r.084	1r.076	+ 0r.008	1r.472	1r.477	− 0r.005
V.	1 .490	1 .481	+ 0 .009	1 .620	1 .626	− 0 .006
VI.	2 .034	2 .045	− 0 .011	1 .881	1 .872	+ 0 .009
VII.	2 .936	2 .936	0 .000	2 .305	2 .305	0 .000
VIII.	4 .679	4 .691	− 0 .012	3 .234	3. 221	+ 0 .013
IX.	12 .532	12 .485	+ 0 .047	7 .560	7 .618	− 0 .058

Metal Gitter of 681 *lines to the mm.*

	Distance $D_1 - D_2$.			Distance of successive Ghosts.		
Order.	Obs.	Calc.	O. — C.	Obs.	Calc.	O. — C.
I.	$0^r.470$	$0^r.470$	$0^r.000$	$1^r.292$	$1^r.292$	$0^r.000$
II.	1 .143	1 .147	— 0 .004	1 .574	1 .573	+ 0 .001
III.	2 .303	2 .304	— 0 .001	2 .110	2 .109	+ 0 .001

Silvered-glass Gitter of 681 *lines per mm.*

	Distance $D_1 - D_2$.			Distance of successive Ghosts.		
Order.	Obs.	Calc.	O. — C.	Obs.	Calc.	O. — C.
I.	$0^r.481$	$0^r.470$	$+ 0^r.011$	$1^r.291$	$1^r.292$	$— 0^r.001$
II.	1 .062	1 .063	— 0 .001	1 .457	1 .457	0 .000
III.	2 .017	2 .021	— 0 .004	1 .840	1 .838	+ 0 .002
IV.	4 .542	4 .544	— 0 .002	3 .115	3 .113	+ 0 .002

The discrepancies between observation and calculation are, in the case of the observations with the coarse-ruled plate in the 4th to the 7th orders, inclusive, pretty well accounted for by the attractions of neighboring lines. This is shown by the subjoined table. In other cases, there are large discrepancies amounting to 7″, or even more, which cannot be so accounted for, and which vastly exceed the errors of observation. Thus, it will almost invariably be found that the ghosts of D_1 are closer together than those of D_2, and that the distances decrease as m increases algebraically. The measures of the ghosts of C and F indicate a much longer period in the inequality. Some attempts have been made to measure the brilliancy of the ghosts. These only roughly agree with the theory.

Detailed Comparison of Calculation and Observation.

Metal Gitter of 340½ *lines per mm.*

Order IV.

	Obs.	Calc.	O. — C.	
G — 1, D_2	$8^r.241$	$8^r.244$	—.003	
G — 1, D_1	9 .330	9 .320	+.010	Carried toward G 0, D_2.
G 0, D_2	9 .723	9 .721	+.002	
G 0, D_1	10 .800	10 .797	+.003	
G + 1, D_2	11 .187	11 .198	—.011	Carried toward G 0, D_1.
G + 1, D_1	12 .272	12 .274	—.002	

Order V.

$G - 1, D_2$	$7^{r}.847$	$7^{r}.846$	$+.001$	
$G - 1, D_1$	9 .337	9 .327	$+.010$	Carried toward $G\,0, D_2$.
$G\,0, D_2$	9 .466	9 .472	$-.006$	Carried toward $G - 1, D_1$.
$G\,0, D_1$	10 .962	10 .953	$+.009$	Carried toward $G + 1, D_2$.
$G + 1, D_2$	11 .090	11 .098	$-.008$	Carried toward $G\,0, D_1$.
$G + 1, D_1$	12 .575	12 .579	$-.004$	

Order VI.

$G - 1, D_2$	$7^{r}.387$	$7^{r}.388$	$-.001$	
$G\,0, D_2$	9 .265	9 .260	$+.005$	Carried a little toward $G - 1, D_1$.
$G - 1, D_1$	9 .421	9 .433	$-.012$	Carried toward $G\,0, D_2$.
$G + 1, D_2$	11 .152	11 .132	$+.020$	Carried toward $G\,0, D_1$.
$G\,0, D_1$	11 .304	11 .305	$-.001$	Carried a little toward $G + 1, D_2$.
$G + 1, D_1$	13 .173	13 .177	$-.004$	

Order VII.

$G - 1, D_2$	$6^{r}.637$	$6^{r}.646$	$-.009$	Single pointings discordant. Rejecting worst obs. $= 6^{r}.643$.
$G\,0, D_2$	8 .955	8 .951	$+.005$	Carried toward $G - 1, D_2$.
$G - 1, D_1$	9 .595	9 .582	$+.013$	Should be carried toward $G\,0, D_2$.
$G + 1, D_2$	11 .262	11 .256	$+.006$	Carried toward $G\,0, D_1$.
$G\,0, D_1$	11 .876	11 .887	$-.011$	Carried toward $G + 1, D_2$.
$G + 1, D_1$	14 .191	14 .192	$-.001$	

Order VIII.

$G - 1, D_2$	$4^{r}.737$	$4^{r}.771$	$-.034$	
$G\,0, D_2$	8 .002	7 .992	$+.010$	
$G - 1, D_1$	9 .467	9 .462	$+.005$	
$G + 1, D_2$	11 .256	11 .213	$+.043$	No distinct attractions.
$G\,0, D_1$	12 .680	12 .683	$-.003$	
$G + 1, D_1$	15 .885	15 .904	$-.019$	

Order IX.

$G - 1, D_2$	$\bar{6}^{r}.865$	$\bar{6}^{r}.812$	$+.053$
$G\,0, D_2$	4 .281	4 .430	$-.149$
$G - 1, D_1$	9 .403	9 .297	$+.106$
$G + 1, D_2$	12 .075	12 .048	$+.027$
$G\,0, D_1$	16 .977	16 .915	$+.062$
$G + 1, D_1$	24 .435	24 .533	$-.098$

Metal Gitter 681 *lines per mm.*

Order I.

				O. — C. —.012	
G — 2, D_2	7^r.286	7^r.323	—.037	—.049	Noted at the time of obs. extremely uncertain.
G — 2, D_1	7 .799	7 .793	+.006	—.006	
G — 1, D_2	8 .632	8 .615	+.017	+.005	
G — 1, D_1	9 .112	9 .085	+.027	+.015	
G 0, D_2	9 .925	9 .907	+.018	+.006	
G 0, D_1	10 .383	10 .377	+.006	—.006	General attraction toward the middle.
G + 1, D_2	11 .196	11 .199	—.003	—.015	
G + 1, D_1	11 .664	11 .669	—.005	—.017	
G + 2, D_2	12 .496	12 .491	+.005	—.007	
G + 2, D_1	12 .928	12 .961	—.033	—.045	

Order II.

				O. — C. —.004
G — 2, D_2	5^r.312	5^r.331	—.019	—.023
G — 2, D_1	6 .482	6 .478	+.004	—.000
G — 1, D_2	6 .907	6 .904	+.003	—.001
G — 1, D_1	8 .059	8 .051	+.008	+.004
G 0, D_2	8 .477	8 .477	.000	—.004
G 0, D_1	9 .627	9 .624	+.003	—.001
G + 1, D_2	10 .067	10 .050	+.017	+.013
G + 1, D_1	11 .191	11 .197	—.006	—.010
G + 2, D_2	11 .632	11 .623	+.009	+.005
G + 2, D_1	12 .752	12 .770	—.018	—.022

Order III.

G — 2, D_2	4^r.593	4^r.627	—.034
G — 1, D_2	6 .713	6 .736	—.023
G — 2, D_1	6 .896	6 .931	—.035
G 0, D_2	8 .876	8 .845	+.031
G — 1, D_1	9 .053	9 .040	+.013
G + 1, D_2	10 .989	10 .954	+.035
G 0, D_1	11 .205	11 .149	+.056
G + 2, D_2	13 .057	13 .063	—.006
G + 1, D_1	13 .280	13 .258	+.022
G + 2, D_1	15 .308	15 .367	—.059